25. PRODUCT OF THREE VECTORS

SCALAR TRIPLE PRODUCT *The scalar triple product of three vectors* $\vec{a}, \vec{b}, \vec{c}$ *is defined as* $(\vec{a} \times \vec{b}) \cdot \vec{c}$, *and it is denoted by* $[\vec{a}\ \vec{b}\ \vec{c}]$.

Thus $[\vec{a}\ \vec{b}\ \vec{c}] = (\vec{a} \times \vec{b}) \cdot \vec{c}$.

Since $(\vec{a} \times \vec{b})$ is a vector, $(\vec{a} \times \vec{b}) \cdot \vec{c}$ is a scalar.

Consequently, $[\vec{a}\ \vec{b}\ \vec{c}]$ is a scalar quantity.

Some Theorems on Scalar Triple Product

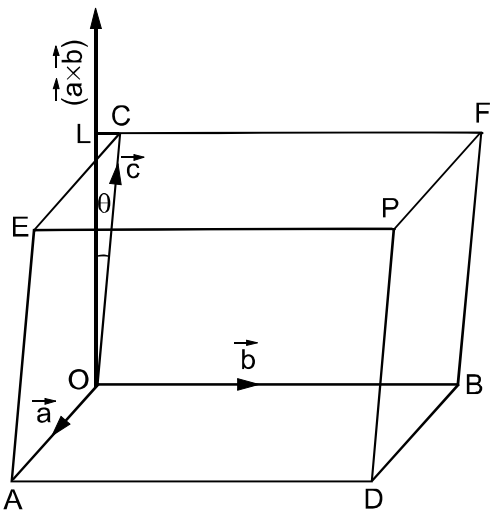
THEOREM 1 (Geometrical interpretation) $(\vec{a} \times \vec{b}) \cdot \vec{c}$ *represents the volume of a parallelepiped whose coterminous edges are represented by* $\vec{a}, \vec{b}, \vec{c}$.

PROOF Let us consider a parallelepiped having coterminous edges OA, OB and OC respectively.

Let $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$ and $\overrightarrow{OC} = \vec{c}$.

Then, $(\vec{a} \times \vec{b})$ is a vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$.

Let θ be the angle between $(\vec{a} \times \vec{b})$ and $\vec{c}$.



$$\therefore \quad [\vec{a}\ \vec{b}\ \vec{c}] = (\vec{a} \times \vec{b}) \cdot \vec{c}.$$

$$= |\vec{a} \times \vec{b}|\,|\vec{c}| \cos\theta$$

$$= (\text{area of } ||\text{gm } OADB)[\text{projection of } \vec{c} \text{ along } (\vec{a} \times \vec{b})]$$

$$= (\text{area of } ||\text{gm } OADB) \cdot OL$$

$$= (\text{area of the base of the parallelepiped}) \cdot (\text{height})$$

$$= \text{volume of the parallelepiped with coterminous edges } \vec{a}, \vec{b}, \vec{c}.$$

Hence $[\vec{a}\ \vec{b}\ \vec{c}]$ represents the volume of the parallelepiped with coterminous edges $\vec{a}, \vec{b}, \vec{c}$ forming a right-handed system.

THEOREM 2 *For any three vectors* $\vec{a}, \vec{b}, \vec{c}$ *prove that* $(\vec{a} \times \vec{b}) \cdot \vec{c} = \vec{a} \cdot (\vec{b} \times \vec{c})$.

PROOF Let $\vec{a}, \vec{b}, \vec{c}$ form a right-handed system, representing the coterminous edges of a parallelepiped of volume V.

Then, $V = (\vec{a} \times \vec{b}) \cdot \vec{c}$.

Again, $\vec{b}, \vec{c}, \vec{a}$ form a right-handed system, representing the coterminous edges of the same parallelepiped.

$\therefore \quad V = (\vec{b} \times \vec{c}) \cdot \vec{a} = \vec{a} \cdot (\vec{b} \times \vec{c}) \quad [\because \ \vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}]$.

Hence, $(\vec{a} \times \vec{b}) \cdot \vec{c} = \vec{a} \cdot (\vec{b} \times \vec{c})$.

THEOREM 3 *The scalar triple product of three vectors remains unchanged so long as their cyclic order remains unchanged,*

i.e., $(\vec{a} \times \vec{b}) \cdot \vec{c} = (\vec{b} \times \vec{c}) \cdot \vec{a} = (\vec{c} \times \vec{a}) \cdot \vec{b}$.

PROOF Let $\vec{a}, \vec{b}, \vec{c}$ form a right-handed system, representing the coterminous edges of a rectangular parallelepiped of volume V.

Then, $V = (\vec{a} \times \vec{b}) \cdot \vec{c}$.

Clearly, $\vec{b}, \vec{c}, \vec{a}$ as well as $\vec{c}, \vec{a}, \vec{b}$ form a right-handed system, representing the coterminous edges of the same parallelepiped.

$\therefore \quad V = (\vec{b} \times \vec{c}) \cdot \vec{a}$ and $V = (\vec{c} \times \vec{a}) \cdot \vec{b}$.

Thus, $(\vec{a} \times \vec{b}) \cdot \vec{c} = (\vec{b} \times \vec{c}) \cdot \vec{a} = (\vec{c} \times \vec{a}) \cdot \vec{b}$ [each equal to V].

Hence, $[\vec{a} \ \vec{b} \ \vec{c}] = [\vec{b} \ \vec{c} \ \vec{a}] = [\vec{c} \ \vec{a} \ \vec{b}]$.

THEOREM 4 *The scalar triple product changes in sign but not in magnitude when the cyclic order of vectors is changed.*

PROOF For any three vectors $\vec{a}, \vec{b}, \vec{c}$, we know that

$$[\vec{a} \ \vec{b} \ \vec{c}] = [\vec{b} \ \vec{c} \ \vec{a}] = [\vec{c} \ \vec{a} \ \vec{b}].$$

$\therefore \quad [\vec{c} \ \vec{b} \ \vec{a}] = (\vec{c} \times \vec{b}) \cdot \vec{a} = -(\vec{b} \times \vec{c}) \cdot \vec{a} = -[\vec{b} \ \vec{c} \ \vec{a}] = -[\vec{a} \ \vec{b} \ \vec{c}]$.

And, $[\vec{a} \ \vec{c} \ \vec{b}] = (\vec{a} \times \vec{c}) \cdot \vec{b} = -(\vec{c} \times \vec{a}) \cdot \vec{b} = -[\vec{c} \ \vec{a} \ \vec{b}] = -[\vec{a} \ \vec{b} \ \vec{c}]$.

$\therefore \quad [\vec{c} \ \vec{b} \ \vec{a}] = -[\vec{a} \ \vec{b} \ \vec{c}]$ and $[\vec{a} \ \vec{c} \ \vec{b}] = -[\vec{a} \ \vec{b} \ \vec{c}]$.

THEOREM 5 *The scalar triple product vanishes if any two of its vectors are equal, i.e.,* $[\vec{a} \ \vec{a} \ \vec{b}] = 0$, $[\vec{a} \ \vec{b} \ \vec{a}] = 0$ *and* $[\vec{b} \ \vec{a} \ \vec{a}] = 0$.

PROOF We have $[\vec{a} \ \vec{a} \ \vec{b}] = (\vec{a} \times \vec{a}) \cdot \vec{b} = \vec{0} \cdot \vec{b} = 0$;

$$[\vec{a}\ \ \vec{b}\ \ \vec{a}] = [\vec{a}\ \ \vec{a}\ \ \vec{b}] \quad \text{[cyclic-ordered property]}$$
$$= (\vec{a} \times \vec{a}) \cdot \vec{b} = \vec{0} \cdot \vec{b} = 0;$$

and $[\vec{b}\ \ \vec{a}\ \ \vec{a}] = [\vec{a}\ \ \vec{b}\ \ \vec{a}]$ [cyclic-ordered property]
$$= 0.$$

Hence, $[\vec{a}\ \ \vec{a}\ \ \vec{b}] = [\vec{a}\ \ \vec{b}\ \ \vec{a}] = [\vec{b}\ \ \vec{a}\ \ \vec{a}] = 0.$

THEOREM 6 *The scalar triple product vanishes if any two of its vectors are parallel or collinear.*

PROOF Let $\vec{a}, \vec{b}, \vec{c}$ be three vectors such that $\vec{a} \parallel \vec{b}$, or $\vec{a}$ and $\vec{b}$ are collinear.

Then, $\vec{a} = m\,\vec{b}$ for some scalar m.

$\therefore \quad [\vec{a}\ \ \vec{b}\ \ \vec{c}] = [m\,\vec{b}\ \ \vec{b}\ \ \vec{c}] = (m\,\vec{b} \times \vec{b}) \cdot \vec{c} = \vec{0} \cdot \vec{c} = 0.$

THEOREM 7 (Scalar triple product in terms of components)

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$
and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$.

Then, $[\vec{a}\ \ \vec{b}\ \ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

PROOF We have $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

$$= (a_2b_3 - a_3b_2)\hat{i} + (a_3b_1 - a_1b_3)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}.$$

$\therefore \quad [\vec{a}\ \ \vec{b}\ \ \vec{c}] = (\vec{a} \times \vec{b}) \cdot \vec{c}$
$$= (a_2b_3 - a_3b_2)c_1 + (a_3b_1 - a_1b_3)c_2 + (a_1b_2 - a_2b_1)c_3$$
$$= a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1)$$
$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

THEOREM 8 *For any three vectors* $\vec{a}, \vec{b}$ *and* $\vec{c}$, *prove that*

$$[\vec{a} + \vec{b},\ \vec{b} + \vec{c},\ \vec{c} + \vec{a}] = 2[\vec{a},\ \vec{b},\ \vec{c}]$$ **[CBSE 2014]**

PROOF We have

$$[\vec{a}+\vec{b}\quad \vec{b}+\vec{c}\quad \vec{c}+\vec{a}]$$

$$=(\vec{a}+\vec{b})\cdot[(\vec{b}+\vec{c})\times(\vec{c}+\vec{a})]$$

$$=(\vec{a}+\vec{b})\cdot[\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{c}+\vec{c}\times\vec{a}] \text{ [by the distributive law]}$$

$$=(\vec{a}+\vec{b})\cdot[(\vec{b}\times\vec{c})-(\vec{a}\times\vec{b})+(\vec{c}\times\vec{a})]$$

$$[\because \vec{c}\times\vec{c}=\vec{0}, \text{ and } \vec{b}\times\vec{a}=-\vec{a}\times\vec{b}]$$

$$=\vec{a}\cdot(\vec{b}\times\vec{c})-\vec{a}\cdot(\vec{a}\times\vec{b})+\vec{a}\cdot(\vec{c}\times\vec{a})+\vec{b}\cdot(\vec{b}\times\vec{c})$$

$$-\vec{b}\cdot(\vec{a}\times\vec{b})+\vec{b}\cdot(\vec{c}\times\vec{a}) \text{ [by the distributive law]}$$

$$=[\vec{a}\ \vec{b}\ \vec{c}]-[\vec{a}\ \vec{a}\ \vec{b}]+[\vec{a}\ \vec{c}\ \vec{a}]+[\vec{b}\ \vec{b}\ \vec{c}]-[\vec{b}\ \vec{a}\ \vec{b}]+[\vec{b}\ \vec{c}\ \vec{a}]$$

$$=[\vec{a}\ \vec{b}\ \vec{c}]+[\vec{b}\ \vec{c}\ \vec{a}]$$

$$[\because \text{ scalar triple product with two equal vectors is } 0]$$

$$=2[\vec{a}\ \vec{b}\ \vec{c}] \quad \{\because [\vec{b}\ \vec{c}\ \vec{a}]=[\vec{a}\ \vec{b}\ \vec{c}]\}.$$

Hence, $[\vec{a}+\vec{b}\quad \vec{b}+\vec{c}\quad \vec{c}+\vec{a}]=2[\vec{a}\ \vec{b}\ \vec{c}]$.

THEOREM 9 *The necessary and sufficient condition for three nonzero, noncollinear vectors* $\vec{a}, \vec{b}, \vec{c}$ *to be coplanar is that* $[\vec{a}\ \vec{b}\ \vec{c}]=0$.

PROOF Let $\vec{a}, \vec{b}, \vec{c}$ be three nonzero, noncollinear and coplanar vectors. Then,

$\vec{b}\times\vec{c}$ is perpendicular to the plane of $\vec{b}$ and $\vec{c}$

$$\Rightarrow \quad (\vec{b}\times\vec{c})\perp\vec{a} \qquad [\because \vec{a}, \vec{b}, \vec{c} \text{ are coplanar}]$$

$$\Rightarrow \quad \vec{a}\cdot(\vec{b}\times\vec{c})=0 \Rightarrow [\vec{a}\ \vec{b}\ \vec{c}]=0.$$

Thus, whenever $\vec{a}, \vec{b}, \vec{c}$ are coplanar, we have $[\vec{a}\ \vec{b}\ \vec{c}]=0$.

Conversely, let $\vec{a}, \vec{b}, \vec{c}$ be three nonzero, noncollinear vectors such that $[\vec{a}\ \vec{b}\ \vec{c}]=\vec{0}$. Then,

$$[\vec{a}\ \vec{b}\ \vec{c}]=0 \Rightarrow \vec{a}\cdot(\vec{b}\times\vec{c})=0$$

$$\Rightarrow \vec{a}=\vec{0}, \text{ or } (\vec{b}\times\vec{c})=\vec{0}, \text{ or } (\vec{b}\times\vec{c})\perp\vec{a}$$

$$\Rightarrow (\vec{b}\times\vec{c})=\vec{0}, \text{ or } (\vec{b}\times\vec{c})\perp\vec{a} \qquad [\because \vec{a}\neq\vec{0}]$$

$$\Rightarrow (\vec{b}\times\vec{c})\perp\vec{a}$$

$$[\because \vec{b}\neq\vec{0}, \ \vec{c}\neq\vec{0}, \text{ and } \vec{b}, \vec{c} \text{ are noncollinear} \Rightarrow \vec{b}\times\vec{c}\neq\vec{0}].$$

Thus, $(\vec{b} \times \vec{c})$ is perpendicular to $\vec{a}$.

But, $(\vec{b} \times \vec{c})$ is perpendicular to the plane of $\vec{b}$ and $\vec{c}$.

$\therefore$ $\vec{a}$ must lie in the plane of $\vec{b}$ and $\vec{c}$.

Hence $\vec{a}, \vec{b}, \vec{c}$ must be coplanar.

NOTE: $\vec{a}, \vec{b}, \vec{c}$ are coplanar $\Leftrightarrow [\vec{a}\ \vec{b}\ \vec{c}] = 0$.

THEOREM 10 *For any three vectors $\vec{a}, \vec{b}, \vec{c}$ show that the vectors $(\vec{a} - \vec{b}), (\vec{b} - \vec{c}), (\vec{c} - \vec{a})$ are coplanar.* **[CBSE 2001]**

PROOF We know that

$(\vec{a} - \vec{b}), (\vec{b} - \vec{c}), (\vec{c} - \vec{a})$ are coplanar

$\Leftrightarrow [\vec{a} - \vec{b}\quad \vec{b} - \vec{c}\quad \vec{c} - \vec{a}] = 0$

Now, $[\vec{a} - \vec{b}\quad \vec{b} - \vec{c}\quad \vec{c} - \vec{a}]$

$= (\vec{a} - \vec{b}) \cdot [(\vec{b} - \vec{c}) \times (\vec{c} - \vec{a})]$

$= (\vec{a} - \vec{b}) \cdot [\vec{b} \times \vec{c} - \vec{b} \times \vec{a} - \vec{c} \times \vec{c} + \vec{c} \times \vec{a}]$ [by the distributive law]

$= (\vec{a} - \vec{b}) \cdot [\vec{b} \times \vec{c} + \vec{a} \times \vec{b} + \vec{c} \times \vec{a}]$ [$\because -\vec{b} \times \vec{a} = \vec{a} \times \vec{b}$, and $\vec{c} \times \vec{c} = \vec{0}$]

$= \vec{a} \cdot (\vec{b} \times \vec{c}) + \vec{a} \cdot (\vec{a} \times \vec{b}) + \vec{a} \cdot [\vec{c} \times \vec{a}]$

$\qquad - \vec{b} \cdot (\vec{b} \times \vec{c}) - \vec{b} \cdot (\vec{a} \times \vec{b}) - \vec{b} \cdot (\vec{c} \times \vec{a})$

$= [\vec{a}\ \vec{b}\ \vec{c}] + [\vec{a}\ \vec{a}\ \vec{b}] + [\vec{a}\ \vec{c}\ \vec{a}] - [\vec{b}\ \vec{b}\ \vec{c}] - [\vec{b}\ \vec{a}\ \vec{b}] - [\vec{b}\ \vec{c}\ \vec{a}]$

$= [\vec{a}\ \vec{b}\ \vec{c}] - [\vec{b}\ \vec{c}\ \vec{a}]$ [$\because$ scalar triple product with two equal vectors is 0]

$= [\vec{a}\ \vec{b}\ \vec{c}] - [\vec{a}\ \vec{b}\ \vec{c}] = 0$ $\{\because [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{a}\ \vec{b}\ \vec{c}]\}$.

Hence, $(\vec{a} - \vec{b}), (\vec{b} - \vec{c}), (\vec{c} - \vec{a})$ are coplanar.

THEOREM 11 *Show that the vectors $\vec{a}, \vec{b}, \vec{c}$ are coplanar if and only if $(\vec{a} + \vec{b}), (\vec{b} + \vec{c}), (\vec{c} + \vec{a})$ are coplanar.* **[CBSE 2013C, '14]**

PROOF $(\vec{a} + \vec{b}), (\vec{b} + \vec{c}), (\vec{c} + \vec{a})$ are coplanar

$\Leftrightarrow [\vec{a} + \vec{b}\quad \vec{b} + \vec{c}\quad \vec{c} + \vec{a}] = 0$

$\Leftrightarrow (\vec{a} + \vec{b}) \cdot \{(\vec{b} + \vec{c}) \times (\vec{c} + \vec{a})\} = 0$

$\Leftrightarrow (\vec{a} + \vec{b}) \cdot \{\vec{b} \times \vec{c} + \vec{b} \times \vec{a} + \vec{c} \times \vec{c} + \vec{c} \times \vec{a}\} = 0$

$$\Leftrightarrow (\vec{a}+\vec{b})\cdot\{\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{a}\}=0 \qquad [\because\ \vec{c}\times\vec{c}=\vec{0}\ ]$$

$$\Leftrightarrow \vec{a}\cdot(\vec{b}\times\vec{c})+\vec{a}\cdot(\vec{b}\times\vec{a})+\vec{a}\cdot(\vec{c}\times\vec{a})$$
$$+\vec{b}\cdot(\vec{b}\times\vec{c})+\vec{b}\cdot(\vec{b}\times\vec{a})+\vec{b}\cdot(\vec{c}\times\vec{a})=0$$

$$\Leftrightarrow [\vec{a}\ \ \vec{b}\ \ \vec{c}]+[\vec{a}\ \ \vec{b}\ \ \vec{a}]+[\vec{a}\ \ \vec{c}\ \ \vec{a}]+[\vec{b}\ \ \vec{b}\ \ \vec{c}]$$
$$+[\vec{b}\ \ \vec{b}\ \ \vec{a}]+[\vec{b}\ \ \vec{c}\ \ \vec{a}]=0$$

$$\Leftrightarrow [\vec{a}\ \ \vec{b}\ \ \vec{c}]+[\vec{b}\ \ \vec{c}\ \ \vec{a}]=0$$

$[\because$ scalar triple product with two equal vectors is 0$]$

$$\Leftrightarrow 2[\vec{a}\ \ \vec{b}\ \ \vec{c}]=0 \quad \{\because\ [\vec{b}\ \ \vec{c}\ \ \vec{a}]=[\vec{a}\ \ \vec{b}\ \ \vec{c}]\}$$

$$\Leftrightarrow [\vec{a}\ \ \vec{b}\ \ \vec{c}]=0$$

$$\Leftrightarrow \vec{a},\ \vec{b},\ \vec{c} \text{ are coplanar.}$$

Hence, $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar if and only if $(\vec{a}+\vec{b})$, $(\vec{b}+\vec{c})$ and $(\vec{c}+\vec{a})$ are coplanar.

THEOREM 12 *For any three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, prove that* $[\vec{a}\ \ \vec{b}+\vec{c}\ \ \vec{a}+\vec{b}+\vec{c}]=0$.

PROOF We have

$$[\vec{a}\ \ \vec{b}+\vec{c}\ \ \vec{a}+\vec{b}+\vec{c}]$$
$$=\{\vec{a}\times(\vec{b}+\vec{c})\}\cdot(\vec{a}+\vec{b}+\vec{c})$$
$$=\{(\vec{a}\times\vec{b})+(\vec{a}\times\vec{c})\}\cdot(\vec{a}+\vec{b}+\vec{c}) \text{ [by the distributive law]}$$
$$=(\vec{a}\times\vec{b})\cdot\vec{a}+(\vec{a}\times\vec{b})\cdot\vec{b}+(\vec{a}\times\vec{b})\cdot\vec{c}+(\vec{a}\times\vec{c})\cdot\vec{a}$$
$$+(\vec{a}\times\vec{c})\cdot\vec{b}+(\vec{a}\times\vec{c})\cdot\vec{c} \text{ [by the distributive law]}$$
$$=[\vec{a}\ \ \vec{b}\ \ \vec{a}]+[\vec{a}\ \ \vec{b}\ \ \vec{b}]+[\vec{a}\ \ \vec{b}\ \ \vec{c}]+[\vec{a}\ \ \vec{c}\ \ \vec{a}]$$
$$+[\vec{a}\ \ \vec{c}\ \ \vec{b}]+[\vec{a}\ \ \vec{c}\ \ \vec{c}]$$
$$=[\vec{a}\ \ \vec{b}\ \ \vec{c}]+[\vec{a}\ \ \vec{c}\ \ \vec{b}]$$

$[\because$ scalar triple product with two equal vectors is 0$]$

$$=[\vec{a}\ \ \vec{b}\ \ \vec{c}]-[\vec{a}\ \ \vec{b}\ \ \vec{c}]=0 \quad \{\because\ [\vec{a}\ \ \vec{c}\ \ \vec{b}]=-[\vec{a}\ \ \vec{b}\ \ \vec{c}]\}.$$

Hence, $[\vec{a}\ \ \vec{b}+\vec{c}\ \ \vec{a}+\vec{b}+\vec{c}]=0$.

THEOREM 13 *If $\vec{a}$, $\vec{b}$, $\vec{c}$ are the position vectors of points A, B, C, prove that $(\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a})$ is a vector perpendicular to the plane of triangle ABC.* **[CBSE 2001C]**

PROOF In order to prove the required result, we have to show that $(\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a})$ is perpendicular to each of the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{CA}$.

We have, $\overrightarrow{AB}=(\vec{b}-\vec{a})$, $\overrightarrow{BC}=(\vec{c}-\vec{b})$ and $\overrightarrow{CA}=(\vec{a}-\vec{c})$.

Now, $(\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a})\cdot(\vec{b}-\vec{a})$

$=(\vec{a}\times\vec{b})\cdot(\vec{b}-\vec{a})+(\vec{b}\times\vec{c})\cdot(\vec{b}-\vec{a})+(\vec{c}\times\vec{a})\cdot(\vec{b}-\vec{a})$

$=(\vec{a}\times\vec{b})\cdot\vec{b}-(\vec{a}\times\vec{b})\cdot\vec{a}+(\vec{b}\times\vec{c})\cdot\vec{b}-(\vec{b}\times\vec{c})\cdot\vec{a}$

$+(\vec{c}\times\vec{a})\cdot\vec{b}-(\vec{c}\times\vec{a})\cdot\vec{a}$

$=[\vec{a}\ \vec{b}\ \vec{b}]-[\vec{a}\ \vec{b}\ \vec{a}]+[\vec{b}\ \vec{c}\ \vec{b}]-[\vec{b}\ \vec{c}\ \vec{a}]+[\vec{c}\ \vec{a}\ \vec{b}]-[\vec{c}\ \vec{a}\ \vec{a}]$

$=[\vec{c}\ \vec{a}\ \vec{b}]-[\vec{b}\ \vec{c}\ \vec{a}]$

$[\because$ scalar triple product with two equal vectors is 0$]$

$=0$ $(\because\ [\vec{c}\ \vec{a}\ \vec{b}]=[\vec{b}\ \vec{c}\ \vec{a}])$.

Similarly, $(\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a})\cdot(\vec{c}-\vec{b})=0$.

And, $(\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a})\cdot(\vec{a}-\vec{c})=0$.

Thus, $(\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a})$ is perpendicular to each one of the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{CA}$, and therefore, it is perpendicular to the plane of $\triangle ABC$.

SOLVED EXAMPLES

EXAMPLE 1 *Prove that* $[\hat{i}\ \hat{j}\ \hat{k}]=1$, *and* $[\hat{i}\ \hat{k}\ \hat{j}]=-1$.

SOLUTION We have

$[\hat{i}\ \hat{j}\ \hat{k}]=(\hat{i}\times\hat{j})\cdot\hat{k}=\hat{k}\cdot\hat{k}=1$ and

$[\hat{i}\ \hat{k}\ \hat{j}]=(\hat{i}\times\hat{k})\cdot\hat{j}=-\hat{j}\cdot\hat{j}=-1$.

EXAMPLE 2 *If* $\vec{a}=2\hat{i}+\hat{j}+3\hat{k}$, $\vec{b}=-\hat{i}+2\hat{j}+\hat{k}$, *and* $\vec{c}=-3\hat{i}+\hat{j}+2\hat{k}$, *find* $[\vec{a}\ \vec{b}\ \vec{c}]$.

SOLUTION We have

$$[\vec{a}\ \vec{b}\ \vec{c}]=\begin{vmatrix}2&1&3\\-1&2&1\\-3&1&2\end{vmatrix}=\begin{vmatrix}0&5&5\\-1&2&1\\0&-5&-1\end{vmatrix}$$

$[R_1\to R_1+2R_2,\ R_3\to R_3-3R_2]$

$=-(-1)\cdot[-5+25]=20$.

EXAMPLE 3 *Find the volume of the parallelepiped whose coterminous edges are represented by the vectors*

$\vec{a} = 2\hat{i} - 3\hat{j} + \hat{k}$, $\vec{b} = \hat{i} - \hat{j} + 2\hat{k}$ and $\vec{c} = 2\hat{i} + \hat{j} - \hat{k}$.

[CBSE 2000C]

SOLUTION We have

$$[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} 2 & -3 & 1 \\ 1 & -1 & 2 \\ 2 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 0 & -1 & -3 \\ 1 & -1 & 2 \\ 0 & 3 & -5 \end{vmatrix}$$

$[R_1 \to R_1 - 2R_2,$ and $R_3 \to R_3 - 2R_2]$

$= (-1)\cdot(5 + 9) = -14.$

$\therefore$ volume of the parallelepiped

$= |[\vec{a}\ \vec{b}\ \vec{c}]| = |-14| = 14$ cubic units.

EXAMPLE 4 *Show that the vectors*

$\hat{i} - 3\hat{j} + 4\hat{k}$, $2\hat{i} - \hat{j} + 2\hat{k}$ *and* $4\hat{i} - 7\hat{j} + 10\hat{k}$ *are coplanar.*

SOLUTION Let $\vec{a} = \hat{i} - 3\hat{j} + 4\hat{k}$, $\vec{b} = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{c} = 4\hat{i} - 7\hat{j} + 10\hat{k}$.

$$\therefore\ [\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} 1 & -3 & 4 \\ 2 & -1 & 2 \\ 4 & -7 & 10 \end{vmatrix} = \begin{vmatrix} 1 & -3 & 4 \\ 0 & 5 & -6 \\ 0 & 5 & -6 \end{vmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 4R_1 \end{matrix}$$

$= (-30 + 30) = 0.$

Hence, the given vectors are coplanar.

EXAMPLE 5 *Find the value of* λ *so that the vectors*

$\vec{a} = 2\hat{i} - 3\hat{j} + \hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} - 3\hat{k}$ *and* $\vec{c} = \hat{j} + \lambda\hat{k}$ *are coplanar.*

SOLUTION The given vectors will be coplanar if $[\vec{a}\ \vec{b}\ \vec{c}] = 0$.

$$\text{Now, } [\vec{a}\ \vec{b}\ \vec{c}] = 0 \Leftrightarrow \begin{vmatrix} 2 & -3 & 1 \\ 1 & 2 & -3 \\ 0 & 1 & \lambda \end{vmatrix} = 0 \Leftrightarrow \begin{vmatrix} 0 & -7 & 7 \\ 1 & 2 & -3 \\ 0 & 1 & \lambda \end{vmatrix} = 0$$

$[R_1 \to R_1 - 2R_2]$

$\Leftrightarrow (-1)(-7\lambda - 7) = 0 \Leftrightarrow 7\lambda + 7 = 0 \Leftrightarrow \lambda = -1.$

Hence, the given vectors are coplanar when $\lambda = -1$.

EXAMPLE 6 *Show that the four points with position vectors* $(4\hat{i} + 5\hat{j} + \hat{k})$, $(-\hat{j} - \hat{k})$, $(3\hat{i} + 9\hat{j} + 4\hat{k})$ *and* $4(-\hat{i} + \hat{j} + \hat{k})$ *are coplanar.* **[CBSE 2014]**

SOLUTION Let the given points be A, B, C, D respectively.

Points A, B, C, D are coplanar $\Leftrightarrow$ $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ are coplanar

$\Leftrightarrow [\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}] = 0.$

Now, $\overrightarrow{AB} = (\text{p.v. of } B) - (\text{p.v. of } A)$

$= (-\hat{j} - \hat{k}) - (4\hat{i} + 5\hat{j} + \hat{k}) = (-4\hat{i} - 6\hat{j} - 2\hat{k})$

$\overrightarrow{AC} = (\text{p.v. of } C) - (\text{p.v. of } A)$

$= (3\hat{i} + 9\hat{j} + 4\hat{k}) - (4\hat{i} + 5\hat{j} + \hat{k}) = (-\hat{i} + 4\hat{j} + 3\hat{k})$

$\overrightarrow{AD} = (\text{p.v. of } D) - (\text{p.v. of } A)$

$= (-4\hat{i} + 4\hat{j} + 4\hat{k}) - (4\hat{i} + 5\hat{j} + \hat{k}) = (-8\hat{i} - \hat{j} + 3\hat{k}).$

$$\therefore \quad [\overrightarrow{AB}\ \ \overrightarrow{AC}\ \ \overrightarrow{AD}] = \begin{vmatrix} -4 & -6 & -2 \\ -1 & 4 & 3 \\ -8 & -1 & 3 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -22 & -14 \\ -1 & 4 & 3 \\ 0 & -21 & -33 \end{vmatrix} \quad \begin{Bmatrix} R_1 \to R_1 - 4R_2 \\ R_3 \to R_3 - 8R_2 \end{Bmatrix}$$

$$= -(-1)[462 - 462] = 0.$$

$\therefore$ $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ are coplanar.

Hence, the points A, B, C, D are coplanar.

EXAMPLE 7 *Find the value of λ so that the four points with position vectors* $(-6\hat{i} + 3\hat{j} + 2\hat{k}), (3\hat{i} + \lambda\hat{j} + 4\hat{k}), (5\hat{i} + 7\hat{j} + 3\hat{k})$ *and* $(-13\hat{i} + 17\hat{j} - 2\hat{k})$ *are coplanar.* **[CBSE 2000]**

SOLUTION Let the given points be A, B, C, D respectively. Then,

$\overrightarrow{AB} = (\text{p.v. of } B) - (\text{p.v. of } A)$

$= (3\hat{i} + \lambda\hat{j} + 4\hat{k}) - (-6\hat{i} + 3\hat{j} + 2\hat{k})$

$= 9\hat{i} + (\lambda - 3)\hat{j} + 2\hat{k}$

$\overrightarrow{AC} = (\text{p.v. of } C) - (\text{p.v. of } A)$

$= (5\hat{i} + 7\hat{j} + 3\hat{k}) - (-6\hat{i} + 3\hat{j} + 2\hat{k}) = (11\hat{i} + 4\hat{j} + \hat{k})$

$\overrightarrow{AD} = (\text{p.v. of } D) - (\text{p.v. of } A)$

$= (-13\hat{i} + 17\hat{j} - \hat{k}) - (-6\hat{i} + 3\hat{j} + 2\hat{k})$

$= (-7\hat{i} + 14\hat{j} - 3\hat{k})$

Now, A, B, C, D are coplanar

$$\Leftrightarrow [\overrightarrow{AB}\ \ \overrightarrow{AC}\ \ \overrightarrow{AD}] = 0 \Leftrightarrow \begin{vmatrix} 9 & \lambda - 3 & 2 \\ 11 & 4 & 1 \\ -7 & 14 & -3 \end{vmatrix} = 0$$

$\Leftrightarrow \quad 9(-12-14)-(\lambda-3)(-33+7)+2(154+28)=0$

$\Leftrightarrow \quad -234+26\lambda-78+364=0 \Leftrightarrow 26\lambda=-52 \Leftrightarrow \lambda=-2.$

Hence, the required value of λ is -2.

EXAMPLE 8 *Show that the points* $A(-1, 4, -3)$, $B(3, 2, -5)$, $C(-3, 8, -5)$ *and* $D(-3, 2, 1)$ *are coplanar.*

SOLUTION Clearly, the position vectors of A, B, C, D are $(-\hat{i}+4\hat{j}-3\hat{k})$, $(3\hat{i}+2\hat{j}-5\hat{k})$, $(-3\hat{i}+8\hat{j}-5\hat{k})$ and $(-3\hat{i}+2\hat{j}+\hat{k})$ respectively.

$\therefore \quad \overrightarrow{AB} = (\text{p.v. of } B) - (\text{p.v. of } A)$

$= (3\hat{i}+2\hat{j}-5\hat{k})-(-\hat{i}+4\hat{j}-3\hat{k}) = (4\hat{i}-2\hat{j}-2\hat{k})$

$\overrightarrow{AC} = (\text{p.v. of } C) - (\text{p.v. of } A)$

$= (-3\hat{i}+8\hat{j}-5\hat{k})-(-\hat{i}+4\hat{j}-3\hat{k}) = (-2\hat{i}+4\hat{j}-2\hat{k})$

$\overrightarrow{AD} = (\text{p.v. of } D) - (\text{p.v. of } A)$

$= (-3\hat{i}+2\hat{j}+\hat{k})-(-\hat{i}+4\hat{j}-3\hat{k}) = (-2\hat{i}-2\hat{j}+4\hat{k}).$

$\therefore \quad [\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}] = \begin{vmatrix} 4 & -2 & -2 \\ -2 & 4 & -2 \\ -2 & -2 & 4 \end{vmatrix} = 2\times(-2)\times(-2)\cdot\begin{vmatrix} 2 & -1 & -1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{vmatrix}$

$= 8\cdot\begin{vmatrix} 0 & -3 & 3 \\ 0 & -3 & 3 \\ 1 & 1 & -2 \end{vmatrix}$

$[R_1 \to R_1 - 2R_3,\ R_2 \to R_2 - R_3]$

$= (8\times 0) = 0 \qquad [\because\ R_1 \text{ and } R_2 \text{ are identical}]$

$\Rightarrow \quad \overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}$ are coplanar

$\Rightarrow$ the points A, B, C, D are coplanar.

EXERCISE 25A

1. Prove that

(i) $[\hat{i}\ \hat{j}\ \hat{k}] = [\hat{j}\ \hat{k}\ \hat{i}] = [\hat{k}\ \hat{j}\ \hat{i}] = 1$

(ii) $[\hat{i}\ \hat{k}\ \hat{j}] = [\hat{k}\ \hat{j}\ \hat{i}] = [\hat{j}\ \hat{i}\ \hat{k}] = 1$

2. Find $[\vec{a}\ \vec{b}\ \vec{c}]$, when

(i) $\vec{a} = 2\hat{i}+\hat{j}+3\hat{k}$, $\vec{b} = -\hat{i}+2\hat{j}+\hat{k}$ and $\vec{c} = 3\hat{i}+\hat{j}+2\hat{k}$

(ii) $\vec{a} = 2\hat{i}-3\hat{j}+4\hat{k}$, $\vec{b} = \hat{i}+2\hat{j}-\hat{k}$ and $\vec{c} = 3\hat{i}-\hat{j}+2\hat{k}$

(iii) $\vec{a} = 2\hat{i} - 3\hat{j}$, $\vec{b} = \hat{i} + \hat{j} - \hat{k}$ and $\vec{c} = 3\hat{i} - \hat{k}$

3. Find the volume of the parallelepiped whose coterminous edges are represented by the vectors

(i) $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = \hat{i} - \hat{j} + \hat{k}$, $\vec{c} = \hat{i} + 2\hat{j} - \hat{k}$

(ii) $\vec{a} = -3\hat{i} + 7\hat{j} + 5\hat{k}$, $\vec{b} = -5\hat{i} + 7\hat{j} - 3\hat{k}$, $\vec{c} = 7\hat{i} - 5\hat{j} - 3\hat{k}$

(iii) $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$, $\vec{b} = 2\hat{i} + \hat{j} - \hat{k}$, $\vec{c} = \hat{j} + \hat{k}$

(iv) $\vec{a} = 6\hat{i}$, $\vec{b} = 2\hat{j}$, $\vec{c} = 5\hat{i}$

4. Show that the vectors $\vec{a}, \vec{b}, \vec{c}$ are coplanar, when

(i) $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$, $\vec{b} = -2\hat{i} + 3\hat{j} - 4\hat{k}$ and $\vec{c} = \hat{i} - 3\hat{j} + 5\hat{k}$

(ii) $\vec{a} = \hat{i} + 3\hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} - \hat{j} - \hat{k}$ and $\vec{c} = 7\hat{j} + 3\hat{k}$

(iii) $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} - 3\hat{k}$ and $\vec{c} = 3\hat{i} - 4\hat{j} + 7\hat{k}$

5. Find the value of λ for which the vectors $\vec{a}, \vec{b}, \vec{c}$ are coplanar, where

(i) $\vec{a} = (2\hat{i} - \hat{j} + \hat{k})$, $\vec{b} = (\hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{c} = (3\hat{i} + \lambda\hat{j} + 5\hat{k})$ **[CBSE 2004]**

(ii) $\vec{a} = \lambda\hat{i} - 10\hat{j} - 5\hat{k}$, $\vec{b} = -7\hat{i} - 5\hat{j}$ and $\vec{c} = \hat{i} - 4\hat{j} - 3\hat{k}$

(iii) $\vec{a} = \hat{i} - \hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} + \hat{j} - \hat{k}$ and $\vec{c} = \lambda\hat{i} - \hat{j} + \lambda\hat{k}$ **[CBSE 2004C]**

6. If $\vec{a} = (2\hat{i} - \hat{j} + \hat{k})$, $\vec{b} = (\hat{i} - 3\hat{j} - 5\hat{k})$ and $\vec{c} = (3\hat{i} - 4\hat{j} - \hat{k})$, find $[\vec{a}\ \vec{b}\ \vec{c}]$ and interpret the result.

7. The volume of the parallelepiped whose edges are $(-12\hat{i} + \lambda\hat{k})$, $(3\hat{j} - \hat{k})$ and $(2\hat{i} + \hat{j} - 15\hat{k})$ is 546 cubic units. Find the value of λ. **[CBSE 2004]**

8. Show that the vectors $\vec{a} = (\hat{i} + 3\hat{j} + \hat{k})$, $\vec{b} = (2\hat{i} - \hat{j} - \hat{k})$ and $\vec{c} = (7\hat{j} + 3\hat{k})$ are parallel to the same plane. {**HINT:** Show that $[\vec{a}\ \vec{b}\ \vec{c}] = 0$}

9. If the vectors $(a\hat{i} + a\hat{j} + c\hat{k})$, $(\hat{i} + \hat{k})$ and $(c\hat{i} + c\hat{j} + b\hat{k})$ be coplanar, show that $c^2 = ab$. **[CBSE 2005]**

10. Show that the four points with position vectors $(4\hat{i} + 8\hat{j} + 12\hat{k})$, $(2\hat{i} + 4\hat{j} + 6\hat{k})$, $(3\hat{i} + 5\hat{j} + 4\hat{k})$ and $(5\hat{i} + 8\hat{j} + 5\hat{k})$ are coplanar.

11. Show that the four points with position vectors $(6\hat{i}-7\hat{j})$, $(16\hat{i}-19\hat{j}-4\hat{k})$, $(3\hat{j}-6\hat{k})$ and $(2\hat{i}-5\hat{j}+10\hat{k})$ are coplanar.

[CBSE 2004, '05]

12. Find the value of λ for which the four points with position vectors $(\hat{i}+2\hat{j}+3\hat{k})$, $(3\hat{i}-\hat{j}+2\hat{k})$, $(-2\hat{i}+\lambda\hat{j}+\hat{k})$ and $(6\hat{i}-4\hat{j}+2\hat{k})$ are coplanar. **[CBSE 2000]**

13. Find the value of λ for which the four points with position vectors $(-\hat{j}+\hat{k})$, $(2\hat{i}-\hat{j}-\hat{k})$, $(\hat{i}+\lambda\hat{j}+\hat{k})$ and $(3\hat{j}+3\hat{k})$ are coplanar. **[CBSE 2000]**

14. Using vector method, show that the points $A(4, 5, 1)$, $B(0, -1, -1)$, $C(3, 9, 4)$ and $D(-4, 4, 4)$ are coplanar.

15. Find the value of λ for which the points $A(3, 2, 1)$, $B(4, \lambda, 5)$, $C(4, 2, -2)$ and $D(6, 5, -1)$ are coplanar. **[CBSE 2004C]**

ANSWERS (EXERCISE 25A)

2. (i) –10 (ii) –7 (iii) 4

3. (i) 4 cubic units (ii) 264 cubic units (iii) 12 cubic units (iv) 60 cubic units

5. (i) $\lambda=-4$ (ii) $\lambda=-3$ (iii) $\lambda=1$ **6.** 0, the given vectors are coplanar

7. $\lambda=-3$ **12.** $\lambda=3$ **13.** $\lambda=1$ **15.** $\lambda=5$

EXERCISE 25B

Very-Short-Answer Questions

1. If $\vec{a}=x\hat{i}+2\hat{j}-z\hat{k}$ and $\vec{b}=3\hat{i}-y\hat{j}+\hat{k}$ are two equal vectors then $x+y+z=?$ **[CBSE 2013]**

2. Write a unit vector in the direction of the sum of the vectors $\vec{a}=(2\hat{i}+2\hat{j}-5\hat{k})$ and $\vec{b}=(2\hat{i}+\hat{j}-7\hat{k})$. **[CBSE 2014]**

3. Write the value of λ so that the vectors $\vec{a}=(2\hat{i}+\lambda\hat{j}+\hat{k})$ and $\vec{b}=(\hat{i}-2\hat{j}+3\hat{k})$ are perpendicular to each other. **[CBSE 2013C]**

4. Find the value of p for which the vectors $\vec{a}=(3\hat{i}+2\hat{j}+9\hat{k})$ and $\vec{b}=(\hat{i}-2p\hat{j}+3\hat{k})$ are parallel. **[CBSE 2014]**

5. Find the value of λ when the projection of $\vec{a} = (\lambda\hat{i} + \hat{j} + 4\hat{k})$ on $\vec{b} = (2\hat{i} + 6\hat{j} + 3\hat{k})$ is 4 units. **[CBSE 2012]**

6. If $\vec{a}$ and $\vec{b}$ are perpendicular vectors such that $|\vec{a} + \vec{b}| = 13$ and $|\vec{a}| = 5$, find the value of $|\vec{b}|$. **[CBSE 2014]**

7. If $\vec{a}$ is a unit vector such that $(\vec{x} - \vec{a}) \cdot (\vec{x} + \vec{a}) = 15$, find $|\vec{x}|$. **[CBSE 2013]**

8. Find the sum of the vectors $\vec{a} = (\hat{i} - 3\hat{k})$, $\vec{b} = (2\hat{j} - \hat{k})$ and $\vec{c} = (2\hat{i} - 3\hat{j} + 2\hat{k})$. **[CBSE 2012]**

9. Find the sum of the vectors $\vec{a} = (\hat{i} - 2\hat{j})$, $\vec{b} = (2\hat{i} - 3\hat{j})$ and $\vec{c} = (2\hat{i} + 3\hat{k})$. **[CBSE 2012]**

10. Write the projection of the vector $(\hat{i} + \hat{j} + \hat{k})$ along the vector $\hat{j}$. **[CBSE 2014]**

11. Write the projection of the vector $(7\hat{i} + \hat{j} - 4\hat{k})$ on the vector $(2\hat{i} + 6\hat{j} + 3\hat{k})$. **[CBSE 2013C]**

12. Find $\vec{a} \cdot (\vec{b} \times \vec{c})$ when $\vec{a} = (2\hat{i} + \hat{j} + 3\hat{k})$, $\vec{b} = (-\hat{i} + 2\hat{j} + \hat{k})$ and $\vec{c} = (3\hat{i} + \hat{j} + 2\hat{k})$. **[CBSE 2014]**

13. Find a vector in the direction of $(2\hat{i} - 3\hat{j} + 6\hat{k})$ which has magnitude 21 units. **[CBSE 2014]**

14. If $\vec{a} = (2\hat{i} + 2\hat{j} + 3\hat{k})$, $\vec{b} = (-\hat{i} + 2\hat{j} + \hat{k})$ and $\vec{c} = (3\hat{i} + \hat{j})$ are such that $(\vec{a} + \lambda\vec{b})$ is perpendicular to $\vec{c}$ then find the value of λ. **[CBSE 2009C]**

15. Write a vector of magnitude 15 units in the direction of vector $(\hat{i} - 2\hat{j} + 2\hat{k})$. **[CBSE 2010]**

16. If $\vec{a} = (\hat{i} + \hat{j} + \hat{k})$, $\vec{b} = (4\hat{i} - 2\hat{j} + 3\hat{k})$ and $\vec{c} = (\hat{i} - 2\hat{j} + \hat{k})$, find a vector of magnitude 6 units which is parallel to the vector $(2\vec{a} - \vec{b} + 3\vec{c})$. **[CBSE 2010]**

17. Write the projection of the vector $(\hat{i} - \hat{j})$ on the vector $(\hat{i} + \hat{j})$. **[CBSE 2011]**

18. Write the angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitudes $\sqrt{3}$ and 2 respectively having $\vec{a} \cdot \vec{b} = \sqrt{6}$. **[CBSE 2011]**

19. If $\vec{a} = (\hat{i} - 7\hat{j} + 7\hat{k})$ and $\vec{b} = (3\hat{i} - 2\hat{j} + 2\hat{k})$ then find $|\vec{a} \times \vec{b}|$. **[CBSE 2008C]**

20. Find the angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitudes 1 and 2 respectively, when $|\vec{a} \times \vec{b}| = \sqrt{3}$. **[CBSE 2009C]**

21. What conclusion can you draw about vectors $\vec{a}$ and $\vec{b}$ when $\vec{a} \times \vec{b} = \vec{0}$ and $\vec{a} \cdot \vec{b} = 0$?

22. Find the value of λ when the vectors $\vec{a} = (\hat{i} + \lambda\hat{j} + 3\hat{k})$ and $\vec{b} = (3\hat{i} + 2\hat{j} + 9\hat{k})$ are parallel.

23. Write the value of $\hat{i} \cdot (\hat{j} \times \hat{k}) + \hat{j} \cdot (\hat{i} \times \hat{k}) + \hat{k} \cdot (\hat{i} \times \hat{j})$.

24. Find the volume of the parallelepiped whose edges are represented by the vectors $\vec{a} = (2\hat{i} - 3\hat{j} + 4\hat{k})$, $\vec{b} = (\hat{i} + 2\hat{j} - \hat{k})$ and $\vec{c} = (3\hat{i} - 2\hat{j} + 2\hat{k})$.

25. If $\vec{a} = (-2\hat{i} - 2\hat{j} + 4\hat{k})$, $\vec{b} = (-2\hat{i} + 4\hat{j} - 2\hat{k})$ and $\vec{c} = (4\hat{i} - 2\hat{j} - 2\hat{k})$ then prove that $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar.

26. If $\vec{a} = (2\hat{i} + 6\hat{j} + 27\hat{k})$ and $\vec{b} = (\hat{i} + \lambda\hat{j} + \mu\hat{k})$ are such that $\vec{a} \times \vec{b} = \vec{0}$ then find the values of λ and μ.

27. If θ is the angle between $\vec{a}$ and $\vec{b}$, and $|\vec{a} \cdot \vec{b}| = |\vec{a} \times \vec{b}|$ then what is the value of θ?

28. When does $|\vec{a} + \vec{b}| = |\vec{a}| + |\vec{b}|$ hold?

29. Find the direction cosines of a vector which is equally inclined to the x-axis, y-axis and z-axis.

30. If P (1, 5, 4) and Q(4, 1, –2) be the position vectors of two points P and Q, find the direction ratios of $\overrightarrow{PQ}$.

31. Find the direction cosines of the vector $\vec{a} = (\hat{i} + 2\hat{j} + 3\hat{k})$.

32. If $\hat{a}$ and $\hat{b}$ are unit vectors such that $(\hat{a} + \hat{b})$ is a unit vector, what is the angle between $\hat{a}$ and $\hat{b}$?

ANSWERS (EXERCISE 25B)

1. 0 **2.** $\frac{1}{13}(4\hat{i} + 3\hat{j} - 12\hat{k})$ **3.** $\lambda = \frac{5}{2}$ **4.** $p = \frac{-1}{3}$ **5.** $\lambda = 5$ **6.** $|\vec{b}| = 12$

7. $|\vec{x}| = 4$ **8.** $(3\hat{i} - \hat{j} - 2\hat{k})$ **9.** $(5\hat{i} - 5\hat{j} + 3\hat{k})$ **10.** 1 **11.** $\frac{8}{7}$ **12.** –10

13. $(6\hat{i} - 9\hat{j} + 18\hat{k})$ **14.** $\lambda = 8$ **15.** $(5\hat{i} - 10\hat{j} + 10\hat{k})$ **16.** $(2\hat{i} - 4\hat{j} + 4\hat{k})$

17. 0 **18.** $\frac{\pi}{4}$ **19.** $19\sqrt{2}$ **20.** $\frac{\pi}{3}$ **21.** $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ **22.** $\lambda = \frac{2}{3}$

23. 1 **24.** 7 cubic untis **26.** $\lambda = 3, \mu = \frac{27}{2}$ **27.** $\frac{\pi}{4}$

28. $\vec{a}$ and $\vec{b}$ are parallel or collinear **29.** $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$ **30.** 3, –4, –6

31. $\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}$ **32.** $\frac{2\pi}{3}$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 25B)

1. $\vec{a} = \vec{b} \Rightarrow (x\hat{i} + 2\hat{j} - z\hat{k}) - (3\hat{i} - y\hat{j} + \hat{k})$

$\Rightarrow x = 3, -y = 2$ and $-z = 1 \Rightarrow x = 3, y = -2$ and $z = -1$

$\Rightarrow (x + y + z) = [3 + (-2) + (-1)] = 0.$

2. Required unit vector $= \frac{(\vec{a} \times \vec{b})}{|\vec{a} \times \vec{b}|} = \frac{(4\hat{i} + 3\hat{j} - 12\hat{k})}{\sqrt{16 + 9 + 144}} = \frac{1}{13}(4\hat{i} + 3\hat{j} - 12\hat{k}).$

3. $\vec{a} \perp \vec{b} \Leftrightarrow \vec{a} \cdot \vec{b} = 0 \Leftrightarrow (2 - 2\lambda + 3) = 0 \Leftrightarrow 2\lambda = 5 \Leftrightarrow \lambda = \frac{5}{2}.$

4. $\vec{a} \parallel \vec{b} \Leftrightarrow \vec{a} \times \vec{b} = \vec{0}$

$$\Leftrightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & 9 \\ 1 & -2p & 3 \end{vmatrix} = 0 \Leftrightarrow (6 + 18p)\hat{i} - (9 - 9)\hat{j} + (-6p - 2)\hat{k} = \vec{0}$$

$\Leftrightarrow (6 + 18p)\hat{i} + 0 \cdot \hat{j} + (-6p - 2)\hat{k} = \hat{0}$

$\Leftrightarrow 6 + 18p = 0$ and $-6p - 2 = 0 \Leftrightarrow p = \frac{-1}{3}.$

5. Clearly, $\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} = 4 \Rightarrow \frac{(2\lambda + 6 + 12)}{\sqrt{4 + 36 + 9}} = 4 \Rightarrow (2\lambda + 18) = 28 \Rightarrow 2\lambda = 10 \Rightarrow \lambda = 5.$

6. Since $\vec{a}$ and $\vec{b}$ are perpendicular vectors, we have $\vec{a} \cdot \vec{b} = 0.$

Now, $|\vec{a} + \vec{b}|^2 = 169 \Rightarrow (\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) = 169$

$\Rightarrow \vec{a} \cdot \vec{a} + 2\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{b} = 169$

$$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 = 169 \quad [\because \ \vec{a}\cdot\vec{b} = 0]$$

$$\Rightarrow 25 + |\vec{b}|^2 = 169 \Rightarrow |\vec{b}|^2 = 144 \Rightarrow |\vec{b}| = 12.$$

7. $(\vec{x}-\vec{a})\cdot(\vec{x}+\vec{a}) = 15 \Rightarrow |\vec{x}|^2 - |\vec{a}|^2 = 15$

$$\Rightarrow |\vec{x}|^2 - 1 = 15 \Rightarrow |\vec{x}|^2 = 16 \Rightarrow |\vec{x}| = 4 \quad \{\because \ |\vec{a}| = 1\}.$$

8. $(\vec{a}+\vec{b}+\vec{c}) = (1+0+2)\hat{i} + (0+2-3)\hat{j} + [(-3)+(-1)+2]\hat{k}$

$$= (3\hat{i} - \hat{j} - 2\hat{k}).$$

9. $(\vec{a}+\vec{b}+\vec{c}) = (1+2+2)\hat{i} + (-2-3+0)\hat{j} + 3\hat{k} = (5\hat{i} - 5\hat{j} + 3\hat{k}).$

10. Required projection $= \dfrac{(\hat{i}+\hat{j}+\hat{k})\cdot\hat{j}}{|\hat{j}|} = \dfrac{(0+1+0)}{1} = 1.$

11. Required projection $= \dfrac{(7\hat{i}+\hat{j}-4\hat{k})\cdot(2\hat{i}+6\hat{j}+3\hat{k})}{|2\hat{i}+6\hat{j}+3\hat{k}|}$

$$= \frac{(14+6-12)}{\sqrt{2^2+6^2+3^2}} = \frac{8}{\sqrt{49}} = \frac{8}{7}.$$

12. $(\vec{b}\times\vec{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & 1 \\ 3 & 1 & 2 \end{vmatrix} = (3\hat{i} + 5\hat{j} - 7\hat{k})$

$$\therefore \quad \vec{a}\cdot(\vec{b}\times\vec{c}) = (2\hat{i}+\hat{j}+3\hat{k})\cdot(3\hat{i}+5\hat{j}-7\hat{k}) = (6+5-21) = -10.$$

13. Required vector $= \dfrac{21(2\hat{i}-3\hat{j}+6\hat{k})}{\sqrt{2^2+(-3)^2+6^2}} = \dfrac{21(2\hat{i}-3\hat{j}+6\hat{k})}{\sqrt{49}}$

$$= 3(2\hat{i} - 3\hat{j} + 6\hat{k}) = (6\hat{i} - 9\hat{j} + 18\hat{k}).$$

14. $(\vec{a}+\lambda\vec{b}) = (2-\lambda)\hat{i} + (2+2\lambda)\hat{j} + (3+\lambda)\hat{k}$

$$(\vec{a}+\lambda\vec{b})\cdot\vec{c} = 0 \Rightarrow [(2-\lambda)\hat{i} + (2+2\lambda)\hat{j} + (3+\lambda)\hat{k}]\cdot(3\hat{i}+\hat{j}) = 0$$

$$\Rightarrow 3(2-\lambda) + 1\cdot(2+2\lambda) = 0 \Rightarrow 6 - 3\lambda + 2 + 2\lambda = 0 \Rightarrow \lambda = 8.$$

15. Required vector $= \dfrac{15(\hat{i}-2\hat{j}+2\hat{k})}{\sqrt{1^2(-2)^2+2^2}} = 5(\hat{i}-2\hat{j}+2\hat{k}) = (5\hat{i} - 10\hat{j} + 10\hat{k}).$

16. $(2\vec{a} - \vec{b} + 3\vec{c}) = (2\hat{i}+2\hat{j}+2\hat{k}) - (4\hat{i} - 2\hat{j} + 3\hat{k}) + (3\hat{i} - 6\hat{j} + 3\hat{k})$

$$= (\hat{i} - 2\hat{j} + 2\hat{k}).$$

$\therefore$ required vector $= \dfrac{6(\hat{i} - 2\hat{j} + 2\hat{k})}{\sqrt{1^2 + (-2)^2 + 2^2}} = \dfrac{6(\hat{i} - 2\hat{j} + 2\hat{k})}{3} = (2\hat{i} - 4\hat{j} + 4\hat{k})$

17. Required projection $= \dfrac{(\hat{i} - \hat{j}) \cdot (\hat{i} + \hat{j})}{|\hat{i} + \hat{j}|} = \dfrac{(\hat{i} \cdot \hat{i} + \hat{i} \cdot \hat{j} - \hat{j} \cdot \hat{i} - \hat{j} \cdot \hat{j})}{\sqrt{1^2 + 1^2}}$

$= \dfrac{(1 + 0 - 0 - 1)}{\sqrt{2}} = 0.$

18. Let θ be the angle between $\vec{a}$ and $\vec{b}$. Then,

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \frac{\sqrt{6}}{\sqrt{3} \times 2} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3} \times \sqrt{2} \times \sqrt{3}}{6}$$

$$= \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}.$$

19. $(\vec{a} \times \vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -7 & 7 \\ 3 & -2 & 2 \end{vmatrix} = (-14 + 14)\hat{i} - (2 - 21)\hat{j} + (-2 + 21)\hat{k}$

$= (19\hat{j} + 19\hat{k})$

$\therefore \quad |\vec{a} \times \vec{b}| = \sqrt{0^2 + (19)^2 + (19)^2} = \sqrt{2 \times (19)^2} = 19\sqrt{2}.$

20. $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta \Rightarrow 1 \times 2 \times \sin\theta = \sqrt{3} \Rightarrow \sin\theta = \dfrac{\sqrt{3}}{2} \Rightarrow \theta = \dfrac{\pi}{3}.$

21. $(\vec{a} \times \vec{b} = \vec{0}$ and $\vec{a} \cdot \vec{b} = 0)$ only when $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$.

22. Clearly, $\vec{a} \parallel \vec{b} \Leftrightarrow \vec{a} \times \vec{b} = \vec{0}$.

$$(\vec{a} \times \vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & \lambda & 3 \\ 3 & 2 & 9 \end{vmatrix} = (9\lambda - 6)\hat{i} - 0\hat{j} + (2 - 3\lambda)\hat{k}$$

$\therefore \quad (\vec{a} \times \vec{b}) = \vec{0} \Leftrightarrow 9\lambda - 6 = 0$ and $2 - 3\lambda = 0 \Leftrightarrow \lambda = \dfrac{2}{3}.$

23. Clearly, $(\hat{j} \times \hat{k}) = \hat{i}$, $(\hat{i} \times \hat{k}) = -\hat{j}$ and $(\hat{i} \times \hat{j}) = \hat{k}$.

$\therefore \quad \hat{i} \cdot (\hat{j} \times \hat{k}) + \hat{j} \cdot (\hat{i} \times \hat{k}) + \hat{k} \cdot (\hat{i} \times \hat{j}) = (\hat{i} \cdot \hat{i} - \hat{j} \cdot \hat{j} + \hat{k} \cdot \hat{k})$

$= (1 - 1 + 1) = 1.$

24. $[\vec{a} \ \ \vec{b} \ \ \vec{c}] = \begin{vmatrix} 2 & -3 & 4 \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{vmatrix} = 2(4 - 1) + 3(2 + 3) + 4(-1 - 6)$

$= (6 + 15 - 28) = -7.$

$\therefore \quad$ volume of the parallelepiped $= |-7| = 7$ cubic units.

25. $[\vec{a}\ \ \vec{b}\ \ \vec{c}] = \begin{vmatrix} -2 & -2 & 4 \\ -2 & 4 & -2 \\ 4 & -2 & -2 \end{vmatrix} = (-2)(-8-4) + 2(4+8) + 4(4-16)$

$= (24 + 24 - 48) = 0.$

Hence $\vec{a}, \vec{b}, \vec{c}$ are coplanar.

26. $(\vec{a} \times \vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 6 & 27 \\ 1 & \lambda & \mu \end{vmatrix} = (6\mu - 27\lambda)\,\hat{i} - (2\mu - 27)\,\hat{j} + (2\lambda - 6)\,\hat{k}.$

$\therefore \quad (\vec{a} \times \vec{b}) = \vec{0} \Leftrightarrow 2\lambda - 6 = 0$ and $2\mu - 27 = 0 \Leftrightarrow \lambda = 3$ and $\mu = \dfrac{27}{2}.$

27. $|\vec{a} \cdot \vec{b}| = |\vec{a} \times \vec{b}| \Rightarrow |\vec{a}|\,|\vec{b}|\cos 0 = |\vec{a}|\,|\vec{b}|\sin 0 \Rightarrow \tan 0 = 1 \Rightarrow 0 = \dfrac{\pi}{4}.$

28. $|\vec{a} + \vec{b}| = |\vec{a}| + |\vec{b}| \Rightarrow |\vec{a} + \vec{b}|^2 = \{|\vec{a}| + |\vec{b}|\}^2$

$\Rightarrow (\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) = |\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|$

$\Rightarrow \vec{a} \cdot \vec{a} + 2\,\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{b} = |\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|$

$\Rightarrow |\vec{a}|^2 + 2\,\vec{a} \cdot \vec{b} + |\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|$

$\Rightarrow 2(\vec{a} \cdot \vec{b}) = 2(|\vec{a}||\vec{b}|) \Rightarrow \vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|$

$\Rightarrow |\vec{a}||\vec{b}|\cos\theta = |\vec{a}||\vec{b}| \Rightarrow \cos\theta = 1 \Rightarrow \theta = 0°.$

$\therefore$ either $\vec{a}$ and $\vec{b}$ are parallel or collinear.

29. $\cos^2\alpha + \cos^2\alpha + \cos^2\alpha = 1 \Rightarrow 3\cos^2\alpha = 1 \Rightarrow \cos\alpha = \dfrac{1}{\sqrt{3}}.$

So, the direction cosines of the vector are $\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}.$

30. D. R.'s of $\overrightarrow{PQ}$ are $(4-1), (1-5), (-2-4)$, i.e., 3, –4, –6.

31. Direction ratios of $\vec{a}$ are 1, 2, 3.

And, $\sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}.$

$\therefore$ direction cosines of $\vec{a}$ are $\dfrac{1}{\sqrt{14}}, \dfrac{2}{\sqrt{14}}, \dfrac{3}{\sqrt{14}}.$

32. $(\hat{a} + \hat{b}) \cdot (\hat{a} + \hat{b}) = 1 \Rightarrow a^2 + b^2 + 2\,\hat{a} \cdot \hat{b} = 1$

$\Rightarrow 1 + 1 + 2\,\hat{a} \cdot \hat{b} = 1 \Rightarrow 2\,\hat{a} \cdot \hat{b} = -1$

$\Rightarrow \hat{a} \cdot \hat{b} = \dfrac{-1}{2} \Rightarrow |\hat{a}|\,|\hat{b}|\cos 0 = \dfrac{-1}{2}$

$\Rightarrow \cos\theta = \dfrac{-1}{2} \Rightarrow \theta = \dfrac{2\pi}{3}.$

OBJECTIVE QUESTIONS

Mark (✓) against the correct answer in each of the following.

1. A unit vector in the direction of the vector $\vec{a} = (2\hat{i} - 3\hat{j} + 6\hat{k})$ is

   (a) $\left(\hat{i} - \frac{3}{2}\hat{j} + 3\hat{k}\right)$ (b) $\left(\frac{2}{5}\hat{i} - \frac{3}{5}\hat{j} + \frac{6}{5}\hat{k}\right)$

   (c) $\left(\frac{2}{7}\hat{i} - \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}\right)$ (d) none of these

2. The direction cosines of the vector $\vec{a} = (-2\hat{i} + \hat{j} - 5\hat{k})$ are

   (a) –2, 1, –5 (b) $\frac{1}{3}, \frac{-1}{6}, \frac{-5}{6}$

   (c) $\frac{2}{\sqrt{30}}, \frac{1}{\sqrt{30}}, \frac{5}{\sqrt{30}}$ (d) $\frac{-2}{\sqrt{30}}, \frac{1}{\sqrt{30}}, \frac{-5}{\sqrt{30}}$

3. If $A(1, 2, -3)$ and $B(-1, -2, 1)$ are the end points of a vector $\overrightarrow{AB}$ then the direction cosines of $\overrightarrow{AB}$ are

   (a) –2, –4, 4 (b) $\frac{-1}{2}, -1, 1$ (c) $\frac{-1}{3}, \frac{-2}{3}, \frac{2}{3}$ (d) none of these

4. If a vector makes angles α, β and γ with the x-axis, y-axis and z-axis respectviely then the value of $(\sin^2\alpha + \sin^2\beta + \sin^2\gamma)$ is

   (a) 1 (b) 2 (c) 0 (d) 3

5. The vector $(\cos\alpha\ \cos\beta)\hat{i} + (\cos\alpha\ \sin\beta)\hat{j} + (\sin\alpha)\hat{k}$ is a

   (a) null vector (b) unit vector

   (c) a constant vector (d) none of these

6. What is the angle which the vector $(\hat{i} + \hat{j} + \sqrt{2}\hat{k})$ makes with the z-axis?

   (a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{6}$ (d) $\frac{2\pi}{3}$

7. If $\vec{a}$ and $\vec{b}$ are vectors such that $|\vec{a}| = \sqrt{3}$, $|\vec{b}| = 2$ and $\vec{a} \cdot \vec{b} = \sqrt{6}$ then the angle between $\vec{a}$ and $\vec{b}$ is

   (a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) $\frac{2\pi}{3}$

8. If $\vec{a}$ and $\vec{b}$ are two vectors such that $|\vec{a}| = |\vec{b}| = \sqrt{2}$ and $\vec{a} \cdot \vec{b} = -1$ then the angle between $\vec{a}$ and $\vec{b}$ is

   (a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{2\pi}{3}$

9. The angle between the vectors $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + \hat{k}$ is

(a) $\cos^{-1}\frac{5}{7}$ (b) $\cos^{-1}\frac{3}{5}$ (c) $\cos^{-1}\frac{3}{\sqrt{14}}$ (d) none of these

10. If $\vec{a} = (\hat{i} + 2\hat{j} - 3\hat{k})$ and $\vec{b} = (3\hat{i} - \hat{j} + 2\hat{k})$ then the angle between $(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$ is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{2}$ (d) $\frac{2\pi}{3}$

11. If $\vec{a} = (\hat{i} + 2\hat{j} - 3\hat{k})$ and $\vec{b} = (3\hat{i} - \hat{j} + 2\hat{k})$ then the angle between $(2\vec{a} + \vec{b})$ and $(\vec{a} + 2\vec{b})$ is

(a) $\cos^{-1}\left(\frac{21}{40}\right)$ (b) $\cos^{-1}\left(\frac{31}{50}\right)$ (c) $\cos^{-1}\left(\frac{11}{30}\right)$ (d) none of these

12. If $\vec{a} = (2\hat{i} + 4\hat{j} - \hat{k})$ and $\vec{b} = (3\hat{i} - 2\hat{j} + \lambda\hat{k})$ be such that $\vec{a} \perp \vec{b}$ then $\lambda = ?$

(a) 2 (b) –2 (c) 3 (d) –3

13. What is the projection of $\vec{a} = (2\hat{i} - \hat{j} + \hat{k})$ on $\vec{b} = (\hat{i} - 2\hat{j} + \hat{k})$?

(a) $\frac{2}{\sqrt{3}}$ (b) $\frac{4}{\sqrt{5}}$ (c) $\frac{5}{\sqrt{6}}$ (d) none of these

14. If $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$, then

(a) $|\vec{a}| = |\vec{b}|$ (b) $\vec{a} \parallel \vec{b}$ (c) $\vec{a} \perp \vec{b}$ (d) none of these

15. If $\vec{a}$ and $\vec{b}$ are mutually perpendicular unit vectors then $(3\vec{a} + 2\vec{b}) \cdot (5\vec{a} - 6\vec{b}) = ?$

(a) 3 (b) 5 (c) 6 (d) 12

16. If the vectors $\vec{a} = 3\hat{i} + \hat{j} - 2\hat{k}$ and $\vec{b} = \hat{i} + \lambda\hat{j} - 3\hat{k}$ are perpendicular to each other then $\lambda = ?$

(a) –3 (b) –6 (c) –9 (d) –1

17. If θ is the angle between two unit vectors $\hat{a}$ and $\hat{b}$ then $\frac{1}{2}|\hat{a} - \hat{b}| = ?$

(a) $\cos\frac{0}{2}$ (b) $\sin\frac{0}{2}$ (c) $\tan\frac{0}{2}$ (d) none of these

18. If $\vec{a} = (\hat{i} - \hat{j} + 2\hat{k})$ and $\vec{b} = (2\hat{i} + 3\hat{j} - 4\hat{k})$ then $|\vec{a} \times \vec{b}| = ?$

(a) $\sqrt{174}$ (b) $\sqrt{87}$ (c) $\sqrt{93}$ (d) none of these

19. If $\vec{a} = (\hat{i} - 2\hat{j} + 3\hat{k})$ and $\vec{b} = (\hat{i} - 3\hat{k})$ then $|\vec{b} \times 2\vec{a}| = ?$

(a) $10\sqrt{3}$ (b) $5\sqrt{17}$ (c) $4\sqrt{19}$ (d) $2\sqrt{23}$

20. If $|\vec{a}| = 2, |\vec{b}| = 7$ and $(\vec{a} \times \vec{b}) = (3\hat{i} + 2\hat{j} + 6\hat{k})$ then the angle between $\vec{a}$ and $\vec{b}$ is

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{2\pi}{3}$ (d) $\frac{3\pi}{4}$

21. If $|\vec{a}| = \sqrt{26}, |\vec{b}| = 7$ and $|\vec{a} \times \vec{b}| = 35$ then $\vec{a} \cdot \vec{b} = ?$

(a) 5 (b) 7 (c) 13 (d) 12

22. Two adjacent sides of a ||gm are represented by the vectors $\vec{a} = (3\hat{i} + \hat{j} + 4\hat{k})$ and $\vec{b} = (\hat{i} - \hat{j} + \hat{k})$. The area of the ||gm is

(a) $\sqrt{42}$ sq units (b) 6 sq units
(c) $\sqrt{35}$ sq units (d) none of these

23. The diagonals of a ||gm are represented by the vectors $\vec{d}_1 = (3\hat{i} + \hat{j} - 2\hat{k})$ and $\vec{d}_2 = (\hat{i} - 3\hat{j} + 4\hat{k})$. The area of the ||gm is

(a) $7\sqrt{3}$ sq units (b) $5\sqrt{3}$ sq units
(c) $3\sqrt{5}$ sq units (d) none of these

24. Two adjacent sides of a triangle are represented by the vectors $\vec{a} = 3\hat{i} + 4\hat{j}$ and $\vec{b} = -5\hat{i} + 7\hat{j}$. The area of the triangle is

(a) 41 sq units (b) 37 sq units (c) $\frac{41}{2}$ sq units (d) none of these

25. The unit vector normal to the plane containing $\vec{a} = (\hat{i} - \hat{j} - \hat{k})$ and $\vec{b} = (\hat{i} + \hat{j} + \hat{k})$ is

(a) $(\hat{j} - \hat{k})$ (b) $(-\hat{j} + \hat{k})$ (c) $\frac{1}{\sqrt{2}}(-\hat{j} + \hat{k})$ (d) $\frac{1}{\sqrt{2}}(-\hat{i} + \hat{k})$

26. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are unit vectors such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ then $(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) = ?$

(a) $\frac{1}{2}$ (b) $\frac{-1}{2}$ (c) $\frac{3}{2}$ (d) $\frac{-3}{2}$

27. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are mutually perpendicular unit vectors then $|\vec{a} + \vec{b} + \vec{c}| = ?$

(a) 1 (b) $\sqrt{2}$ (c) $\sqrt{3}$ (d) 2

28. $[\hat{i}\ \ \hat{j}\ \ \hat{k}] = ?$

(a) 0 (b) 1 (c) 2 (d) 3

29. If $\vec{a} = (2\hat{i} - 3\hat{j} + 4\hat{k})$, $\vec{b} = (\hat{i} + 2\hat{j} - \hat{k})$ and $\vec{c} = (3\hat{i} - \hat{j} - 2\hat{k})$ be the coterminous edges of a parallelepiped then its volume is

(a) 21 cubic units (b) 14 cubic units

(c) 7 cubic units (d) none of these

30. If the volume of a parallelepiped having $\vec{a} = (5\hat{i} - 4\hat{j} + \hat{k})$, $\vec{b} = (4\hat{i} + 3\hat{j} + \lambda\hat{k})$ and $\vec{c} = (\hat{i} - 2\hat{j} + 7\hat{k})$ as coterrminous edges, is 216 cubic units then the value of λ is

(a) $\frac{5}{3}$ (b) $\frac{4}{3}$ (c) $\frac{2}{3}$ (d) $\frac{1}{3}$

31. It is given that the vectors $\vec{a} = (2\hat{i} - 2\hat{k})$, $\vec{b} = \hat{i} + (\lambda + 1)\hat{j}$ and $\vec{c} = (4\hat{i} + 2\hat{k})$ are coplanar. Then, the value of λ is

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) 2 (d) 1

32. Which of the following is meaningless?

(a) $\vec{a} \cdot (\vec{b} \times \vec{c})$ (b) $\vec{a} \times (\vec{b} \cdot \vec{c})$ (c) $(\vec{a} \times \vec{b}) \cdot \vec{c}$ (d) none of these

33. $\vec{a} \cdot (\vec{a} \times \vec{b}) = ?$

(a) 0 (b) 1 (c) $a^2 b$ (d) meaningless

34. For any three vectors $\vec{a}, \vec{b}, \vec{c}$ the value of $[\vec{a} - \vec{b}\ \ \vec{b} - \vec{c}\ \ \vec{c} - \vec{a}]$ is

(a) $2[\vec{a}\ \ \vec{b}\ \ \vec{c}]$ (b) 1 (c) 0 (d) none of these

ANSWERS (OBJECTIVE QUESTIONS)

1. (c)	**2.** (d)	**3.** (c)	**4.** (b)	**5.** (b)	**6.** (a)	**7.** (c)
8. (d)	**9.** (a)	**10.** (c)	**11.** (b)	**12.** (b)	**13.** (c)	**14.** (c)
15. (a)	**16.** (c)	**17.** (b)	**18.** (c)	**19.** (c)	**20.** (a)	**21.** (b)
22. (a)	**23.** (b)	**24.** (c)	**25.** (c)	**26.** (d)	**27.** (c)	**28.** (b)
29. (c)	**30.** (a)	**31.** (d)	**32.** (b)	**33.** (a)	**34.** (c)	

HINTS TO THE GIVEN OBJECTIVE QUESTIONS

1. $|\vec{a}| = \sqrt{2^2 + (-3)^2 + 6^2} = \sqrt{49} = 7.$

$\therefore$ required unit vector is $\left(\frac{2}{7}\hat{i} - \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}\right)$.

2. $|\vec{a}| = \sqrt{(-2)^2 + 1^2 + (-5)^2} = \sqrt{30}.$

$\therefore$ direction cosines of $\vec{a}$ are $\frac{-2}{\sqrt{30}}, \frac{1}{\sqrt{30}}, \frac{-5}{\sqrt{30}}$.

3. $\overrightarrow{AB}$ = (p.v. of B) – (p.v. of A)

$= (-\hat{i} - 2\hat{j} + \hat{k}) - (\hat{i} + 2\hat{j} - 3\hat{k}) = (-2\hat{i} - 4\hat{j} + 4\hat{k}).$

$|\overrightarrow{AB}| = \sqrt{(-2)^2 + (-4)^2 + 4^2} = \sqrt{36} = 6.$

$\therefore$ direction cosines of $\overrightarrow{AB}$ are $\frac{-2}{6}, \frac{-4}{6}, \frac{4}{6}$, i.e., $\frac{-1}{3}, \frac{-2}{3}, \frac{2}{3}$.

4. We have $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\Rightarrow (1 - \sin^2\alpha) + (1 - \sin^2\beta) + (1 - \sin^2\gamma) = 1$

$\Rightarrow (\sin^2\alpha + \sin^2\beta + \sin^2\gamma) = 2.$

5. Let $\vec{a} = (\cos\alpha\cos\beta)\hat{i} + (\cos\alpha\sin\beta)\hat{j} + (\sin\alpha)\hat{k}$. Then,

$|\vec{a}|^2 = (\cos\alpha\cos\beta)^2 + (\cos\alpha\sin\beta)^2 + (\sin\alpha)^2$

$= \cos^2\alpha\cos^2\beta + \cos^2\alpha\sin^2\beta + \sin^2\alpha$

$= \cos^2\alpha(\cos^2\beta + \sin^2\beta) + \sin^2\alpha$

$= (\cos^2\alpha + \sin^2\alpha) = 1.$

Hence, the given vector $\vec{a}$ is a unit vector.

6. $\vec{a} = (\hat{i} + \hat{j} + \sqrt{2}\hat{k}) \Rightarrow |\vec{a}| = \sqrt{1^2 + 1^2 + (\sqrt{2})^2} = \sqrt{4} = 2.$

Direction ratios of $\vec{a}$ are $1, 1, \sqrt{2}$.

Direction cosines of $\vec{a}$ are $\frac{1}{2}, \frac{1}{2}, \frac{\sqrt{2}}{2}$, i.e., $\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}$.

$\therefore \cos\gamma = \frac{1}{\sqrt{2}} \Rightarrow \gamma = \frac{\pi}{4}.$

7. $\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{\sqrt{6}}{\sqrt{3}\times 2} = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}.$

8. $\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{-1}{\sqrt{2}\times\sqrt{2}} = \frac{-1}{2} \Rightarrow \theta = \frac{2\pi}{3}.$

9. $\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{[1\times 3 + (-2)\times(-2) + 3\times 1]}{[\sqrt{1^2 + (-2)^2 + 3^2}][\sqrt{3^2 + (-2)^2 + 1^2}]} = \frac{10}{14} = \frac{5}{7} \Rightarrow \theta = \cos^{-1}\frac{5}{7}.$

10. $(\vec{a} + \vec{b}) = (4\hat{i} + \hat{j} - \hat{k})$ and $(\vec{a} - \vec{b}) = (-2\hat{i} + 3\hat{j} - 5\hat{k}).$

$\therefore \cos\theta = \frac{(\vec{a} + \vec{b})\cdot(\vec{a} - \vec{b})}{|\vec{a} + \vec{b}|\cdot|\vec{a} - \vec{b}|} = \frac{-8 + 3 + 5}{\sqrt{18}\times\sqrt{38}} = 0 \Rightarrow \theta = \frac{\pi}{2}.$

11. $(2\vec{a}+\vec{b})=(5\hat{i}+3\hat{j}-4\hat{k})$ and $(\vec{a}+2\vec{b})=(7\hat{i}+0\hat{j}+\hat{k})$.

$\therefore \quad \cos\theta=\dfrac{(5\times7+3\times0-4\times1)}{\sqrt{50}\times\sqrt{50}}=\dfrac{31}{50}\Rightarrow\ \theta=\cos^{-1}\left(\dfrac{31}{50}\right)$.

12. $\vec{a}\perp\vec{b}\Rightarrow\ \vec{a}\cdot\vec{b}=0\Rightarrow\ 2\times3+4\times(-2)+(-1)\times\lambda=0$

$\Rightarrow\ \lambda=(6-8)=-2.$

13. Projection of $\vec{a}$ on $\vec{b}=\dfrac{(\vec{a}\cdot\vec{b})}{|\vec{a}|}=\dfrac{2\times1+(-1)\times(-2)+1\times1}{\sqrt{1^2+(-2)^2+1^2}}=\dfrac{5}{\sqrt{6}}$.

14. $|\vec{a}+\vec{b}|=|\vec{a}-\vec{b}|\Rightarrow|\vec{a}+\vec{b}|^2=|\vec{a}-\vec{b}|^2$

$\Rightarrow\ (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})=(\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b})$

$\Rightarrow\ 4(\vec{a}\cdot\vec{b})=0\Rightarrow\ \vec{a}\cdot\vec{b}=0\Rightarrow\ \vec{a}\perp\vec{b}$.

15. Given $a^2=b^2=1$ and $\vec{a}\cdot\vec{b}=0$.

$\therefore \quad (3\vec{a}+2\vec{b})\cdot(5\vec{a}-6\vec{b})=15a^2-12b^2-8\vec{a}\cdot\vec{b}$

$=(15\times1)-(12\times1)-(8\times0)$

$=(15-12-0)=3.$

16. $\vec{a}\perp\vec{b}\Rightarrow\ \vec{a}\cdot\vec{b}=0$

$\Rightarrow\ 3\times1+1\times\lambda+(-2)\times(-3)=0\Rightarrow\ \lambda=-9.$

17. $|\hat{a}-\hat{b}|^2=(\hat{a}-\hat{b})\cdot(\hat{a}-\hat{b})=\hat{a}\cdot\hat{a}+\hat{b}\cdot\hat{b}-2\hat{a}\cdot\hat{b}$

$=(1+1-2\times1\times1\times\cos\theta)=2(1-\cos\theta)=4\sin^2\dfrac{\theta}{2}$.

$\therefore \quad \sin^2\dfrac{\theta}{2}=\dfrac{1}{4}\cdot|\hat{a}-\hat{b}|^2\Rightarrow\ \dfrac{1}{2}|\hat{a}-\hat{b}|=\sin\dfrac{\theta}{2}$.

18. $(\vec{a}\times\vec{b})=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\1&-1&2\\2&3&-4\end{vmatrix}=(4-6)\hat{i}-(-4-4)\hat{j}+(3+2)\hat{k}$

$=(-2)\hat{i}+8\hat{j}+5\hat{k}.$

$\therefore \quad |\vec{a}\times\vec{b}|^2=\{(-2)^2+8^2+5^2\}=(4+64+25)=93\Rightarrow\ |\vec{a}\times\vec{b}|=\sqrt{93}.$

19. $\vec{b}=(\hat{i}+0\hat{j}-3\hat{k})$ and $2\vec{a}=(2\hat{i}-4\hat{j}+6\hat{k})$.

$\therefore \quad (\vec{b}\times2\vec{a})=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\1&0&-3\\2&-4&6\end{vmatrix}=(0-12)\hat{i}-(6+6)\hat{j}+(-4-0)\hat{k}$

$=(-12\hat{i}-12\hat{j}-4\hat{k}).$

$\therefore \quad |\vec{b} \times 2\vec{a}|^2 = \{(-12)^2 + (-12)^2 + (-4)^2\} = (144 + 144 + 16) = 304.$

Hence, $|\vec{b} \times 2\vec{a}| = \sqrt{304} = 4\sqrt{19}.$

20. $|\vec{a} \times \vec{b}| = \sqrt{3^2 + 2^2 + 6^2} = \sqrt{49} = 7$

$\Rightarrow \quad |\vec{a}||\vec{b}|\sin\theta = 7 \Rightarrow \sin\theta = \frac{7}{2 \times 7} = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}.$

21. $|\vec{a} \times \vec{b}| = 35 \Rightarrow |\vec{a}||\vec{b}|\sin\theta = 35 \Rightarrow \sin\theta = \frac{35}{\sqrt{26} \times 7} = \frac{5}{\sqrt{26}}.$

$\therefore \quad \cos\theta = \sqrt{1 - \sin^2\theta} = \sqrt{1 - \frac{25}{26}} = \frac{1}{\sqrt{26}}.$

$\therefore \quad \vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta = \sqrt{26} \times 7 \times \frac{1}{\sqrt{26}} = 7.$

22. $(\vec{a} \times \vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & 4 \\ 1 & -1 & 1 \end{vmatrix} = (1+4)\hat{i} - (3-4)\hat{j} + (-3-1)\hat{k} = 5\hat{i} + \hat{j} - 4\hat{k}.$

Required area $= |\vec{a} \times \vec{b}| = \sqrt{25 + 1 + 16} = \sqrt{42}$ sq units.

23. $(\vec{d}_1 \times \vec{d}_2) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix} = (4-6)\hat{i} - (12+2)\hat{j} + (-9-1)\hat{k}$

$= (-2\hat{i} - 14\hat{j} - 10\hat{k})$

$\Rightarrow \quad |\vec{d}_1 \times \vec{d}_2| = \sqrt{(-2)^2 + (-14)^2 + (-10)^2} = \sqrt{300} = 10\sqrt{3}$

$\Rightarrow \quad$ area of the $||$gm $= \frac{1}{2}|\vec{d}_1 \times \vec{d}_2| = 5\sqrt{3}$ sq units.

24. $(\vec{a} \times \vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 4 & 0 \\ -5 & 7 & 0 \end{vmatrix} = 0\hat{i} - 0\hat{j} + (21+20)\hat{k}$

$\Rightarrow \Delta = \frac{1}{2}|\vec{a} \times \vec{b}| = \frac{41}{2}$ sq units.

25. $(\vec{a} \times \vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & -1 \\ 1 & 1 & 1 \end{vmatrix} = (-2\hat{j} + 2\hat{k})$

$\Rightarrow |\vec{a} \times \vec{b}| = \sqrt{(-2)^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$

$\Rightarrow \quad$ required vector $= \frac{2(-\hat{j} + \hat{k})}{2\sqrt{2}} = \frac{1}{\sqrt{2}}(-\hat{j} + \hat{k}).$

26. $(\vec{a}+\vec{b}+\vec{c})\cdot(\vec{a}+\vec{b}+\vec{c})=0$

$\Rightarrow |\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=0$

$\Rightarrow \quad (\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=\dfrac{-3}{2} \quad [\because |\vec{a}|=|\vec{b}|=|\vec{c}|=1].$

27. Given $a^2=b^2=c^2=1$ and $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{c}=\vec{c}\cdot\vec{a}=0.$

$\therefore \quad |\vec{a}+\vec{b}+\vec{c}|^2=(\vec{a}+\vec{b}+\vec{c})\cdot(\vec{a}+\vec{b}+\vec{c})$

$=\vec{a}\cdot\vec{a}+\vec{b}\cdot\vec{b}+\vec{c}\cdot\vec{c}+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})$

$=a^2+b^2+c^2+2\times 0=(1+1+1+0)=3.$

Hence, $|\vec{a}+\vec{b}+\vec{c}|=\sqrt{3}.$

28. $[\hat{i}\ \ \hat{j}\ \ \hat{k}]=(\hat{i}\times\hat{j})\cdot\hat{k}=\hat{k}\cdot\hat{k}=1.$

29. We have

$$(\vec{a}\times\vec{b})\cdot\vec{c}=\begin{vmatrix}2 & -3 & 4\\ 1 & 2 & -1\\ 3 & -1 & 2\end{vmatrix}=2(4-1)+3(2+3)+4(-1-6)$$

$=(6+15-28)=-7.$

$\therefore$ volume of the parallelopiped

$=|(\vec{a}\times\vec{b})\cdot\vec{c}|=|-7|=7$ cubic units.

30. $$(\vec{a}\times\vec{b})\cdot\vec{c}=\begin{vmatrix}5 & -4 & 1\\ 4 & 3 & \lambda\\ 1 & -2 & 7\end{vmatrix}=5(21+2\lambda)+4(28-\lambda)+1\cdot(-8-3)$$

$=(105+112-11)+(10\lambda-4\lambda)=(206+6\lambda).$

$\therefore \quad 206+6\lambda=216 \Rightarrow 6\lambda=10 \Rightarrow \lambda=\dfrac{10}{6}=\dfrac{5}{3}.$

31. Since $\vec{a},\vec{b},\vec{c}$ are coplanar, we must have $[\vec{a}\ \ \vec{b}\ \ \vec{c}]=0.$

Now, $$[\vec{a}\ \ \vec{b}\ \ \vec{c}]=\begin{vmatrix}2 & 0 & -2\\ 1 & \lambda+1 & 0\\ 0 & 4 & 2\end{vmatrix}$$

$=2(2\lambda+2-0)-2(4-0)$

$=4\lambda+4-8=4\lambda-4.$

$\therefore \quad [\vec{a}\ \ \vec{b}\ \ \vec{c}]=0 \Leftrightarrow 4\lambda-4=0 \Leftrightarrow 4\lambda=4 \Leftrightarrow \lambda=1.$

32. Clearly, $\vec{a}\times(\vec{b}\cdot\vec{c})$ is meaningless.

33. $\vec{a}\cdot(\vec{a}\times\vec{b})=(\vec{a}\times\vec{a})\cdot\vec{b}=\vec{0}\cdot\vec{b}=0.$

34. We have proved in the text that $[\vec{a}-\vec{b}\ \ \vec{b}-\vec{c}\ \ \vec{c}-\vec{a}]=0.$